A STOCHASTIC MODELING OF A STUDENT'S UNITS COORDINATION ACTIVITY

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In this proposal, I introduce a method for modeling the dynamics of a sixth-grade student's accommodation of his fractions scheme to include a disembedding operation (Steffe & Olive, 2010). I will describe a three-part approach consisting of a constructivist teaching experiment, retrospective analysis, and stochastic modeling of the student's activity across ecologies within the teaching experiment (cf. Steffe & Thompson, 2000). As disembedding requires coordination of two levels of units—a disembedded part and an unadulterated whole (Steffe, 2001)—the results also serve as an initial model for describing the dynamics of an individual's units coordinating within and across contexts.

Keywords: Number Concepts and Operations, Research Methods, Learning Trajectories

Introduction

Researchers employing constructivist teaching experiment methodology (Steffe & Thompson, 2000) have developed a hypothesized hierarchy of fractions schemes; each scheme is individually defined via its associated operational necessities (see Steffe, 2001). This structure of structures has been refined and verified via subsequent teaching experiments (e.g., Hackenberg, 2007; Norton, 2008), and written assessments and clinical interviews have been used to quantitatively verify that the hypothesized sequence of scheme construction generalizes (Norton & Wilkins, 2009; 2010; 2012; Wilkins, Norton, & Boyce, 2013). However, description of the *process* of conceptual change (accommodation and assimilation) *within* the hierarchy of fraction schemes has heretofore been primarily qualitative. In this proposal, I will describe, exemplify, and discuss the utility of a mathematical model for describing the non-linear development of an important operation within the hypothesized hierarchy—the disembedding operation for fractions.

Theoretical Perspectives

Fractions Schemes, Disembedding, and Units Coordination

Building upon von Glasersfeld's (1995) tri-partite notion of scheme (recognition template, coordinated cognitive actions, and expected result), Steffe has detailed a sequence of fraction schemes in terms of learners' reorganization of their operations first constructed for whole numbers (Steffe & Olive, 2010). A pre-cursor to the first genuine fractional scheme (the partitive fraction scheme) is the development of a *disembedding* operation for fractions, in which a part can be removed from a whole without its adulteration. Per Steffe's (2001) reorganization hypothesis, the disembedding operation for fractions is a reorganization of its role in the construction of an explicitly nested number sequence, in which a number like twelve is structured as a single composite unit (twelve) consisting of twelve units (ones).

Paramount to the construction of the disembedding operation for fractions is the ability to coordinate operations with two levels of units – the disembedded part and the whole. Coordinating two levels of units *in activity* involves operating with and preserving the structural relationship of the units with reliance on sensory-motor activity. The *interioriziation* of such a

two level structure (i.e., the development of an ability for it to become part of one's assimilatory or recognition template) has been shown necessary for the construction of more advanced fraction schemes such as the reversible partitive fraction scheme (Hackenberg, 2007; Norton & Wilkins, 2010) and is also necessary for the construction of more advanced multiplicative concepts necessary for algebra (Hackenberg & Lee, 2012; Hackenberg & Tillema, 2009).

Observing Conceptual Change

Conceptual change occurs via a process of *reflective abstraction* (Piaget, 1970), which includes (not necessarily conscious) reflection on mental re-presentations of experiential activity, pattern recognition leading to abstraction, and construction of re-presentations of the activity in which to assimilate future similar experiences. As a researcher's observations are likewise constructed, modeling mathematical learning is best approached via social interaction, i.e., in the process of teaching.

> Just as students contribute mathematics to experiential situations to establish them as mathematical situations, we adults contribute our concepts and observations to what we observe students do. … So, rather than trying to make models of mathematical learning in terms of concepts embedded in the outside world, we, as constructivists, try to build models of paths of interactive mathematical communication, paths that cannot be specified a priori. (Steffe & Wiegel, 1996, p. 494).

While not realizable to the researcher before retrospective analysis, the trajectory of a student's schematic development, culminating in inference of *conceptual change,* consists of non-observation, observation in activity, observation of the use of mental records instead of physical action (internalization), and observation of immediacy without apparent conscious operational activity (interiorization). But once a scheme emerges, its successive observation again, in ostensibly the same context, is uncertain (see Tzur, 2007). Even in the most structured of clinical interview settings, describing the context of an activity necessitates reduction of the multitude of possible factors (Clement, 2000). For the purposes of this study, an activity's context is considered to have two dimensions, detailed in the methods section below: that of the task itself and that of the individual learner's environment.

Methods/Methodology

Teaching Experiment

For this proposal, I focus on Charles, who was one of four students that I taught in a six-session paired-student constructivist teaching experiment (Steffe & Thompson, 2000) over three weeks during the summer, between sixth and seventh grade. Each student's first session consisted of a clinical interview for assessing students' fraction schemes (Wilkins, Norton, & Boyce, 2013). My goals for the remaining five sessions were to (1) build and refine a model for each student's mathematics, to include their ways of operating in multiple task contexts (i.e., with circular and linear fraction models) and sequences, and (2) support (to the extent possible in the short period of time) their construction of the operations theorized as necessary for the emergence of more powerful fraction schemes. For planning purposes, the videographer/witness familiar with fraction schemes and I discussed our analysis of the progress of the teaching experiment between sessions.

Transcription Analysis

Retrospective analysis began with a rich transcription of videos. My goal was to document, from my perspective at the conclusion of the teaching experiment, every observed or inferred action by Charles, his co-participant D.J., myself as the teacher-researcher, or the witness, that might reasonably pertain to Charles' coordination of units. Context descriptions were categorized

across two independent frameworks. For social interaction, the framework suggested by Simon et al. (2010) for limited teacher-student interaction was expanded to categorize both teacher and peer actions. Categories for teacher actions included re-focusing, asking for explanation, affirming, reiterating, demonstrating, and summarizing; asking for explanation, demonstrating, and affirming are examples of categories of students' activities. For task context, Wagner's (2006) framework was slightly modified to consist of task *type* (i.e., naming a fraction versus constructing a fractional size), *aspect* (i.e., coordinate units within 8 rather than units within 2), or *setting* (i.e., linear fraction rods versus circular drawings).

Characterizations of context were used to separate Charles' observed activity into *chunks* to be analyzed. A *chunk* is a portion of time during which Charles indicated mathematical activity and the context of that activity (as coded) was unchanged. For example, if Charles responded to a task, and subsequently I asked him to explain his thinking, his second response would be in a separate chunk. Thus, the difference between two chunks is not a quantitative difference in seconds, but rather a qualitative difference in Charles' ecological environment. Since context distinctions were categorized from my perspective of Charles' environment, I attempted to bracket my retrospective hypotheses about how Charles' might have constructed his environment in my coding. However, hypothesizing indirectly affected the results of categorizing by chunk because of its role in influencing my responses and decisions during and between the teaching experiment sessions (Steffe & Thompson, 2000).

Each chunk was dichotomously coded for whether there was an opportunity for units coordination (disembedding) to be inferred. For chunks in which I was able to make an inference on Charles' units coordinating activity, I coded '1' if units coordination was inferred and '0' if I inferred that Charles was not coordinating units. I coded for these inferences based on my model of Charles' mathematics formed at the conclusion of the teaching experiment.

Stochastic Model

I created a model of the dynamics of Charles' units coordination as a derivation of a Markov model proposed by Bush & Mosteller (1953). The objects of analysis for measuring growth are an individual's propensities for changing to or maintaining a "higher level" of thinking across chunks. I use the term *propensities* instead of *probabilities* in order to emphasize that the propensities are not computed as ratios of successful outcomes to total outcomes. Rather, the first simplifying assumption is that the propensity of coordinating units (and not coordinating units) across two chunks is a function of the current inference and the propensities in the preceding chunk. This is because the context of subsequent inferences is affected by the relationships between the propensities, the current observation, and the teacher-researchers' goals, as these relationships affect the teacher-researcher's decisions. For example, upon hypothesizing that Charles seems to be "consistently" coordinating units in activity, I would likely alter the context in order to engage him in further learning, rather than continuing to engage him in a "similar" manner. I address the notion of explaining what is meant by "consistently" in the formulation below, while saving consideration of what is meant by "similar" to future work.

Define $T_i = \begin{pmatrix} p_i & 1-p_i \\ 1-q_i & q_i \end{pmatrix}$, where p_i is the propensity of *not* coordinating units on chunk i, given that units *were not* coordinated in chunk i -1; q_i is the propensity that units *are* coordinated on chunk i, given that units *were* coordinated in chunk i. One can think of p as the conditional (Bayesian) probability P(0|0) and q as P(1|1). Without a priori knowledge, there is no reason to predict that a subsequent chunk's inference will be different or the same as its

predecessor. Hence, it is appropriate to assign initial equipropensity: $p_0 = q_0 = 0.5$. Let $c_i \in \{0,1\}$ be the observation of units coordination during chunk *i*. Algorithm 1 below is my recursive method for computing p_i and q_i. By producing a plot of the values of *p* and *q* by chunk, one can communicate the trajectory of a student's ways of operating in a way amenable to a broad range of analyses.

Algorithm 1. Computing $T_i = \begin{pmatrix} p_i & 1-p_i \\ 1-q_i & q_i \end{pmatrix}, i = 1,2,3,...$

If $c_{i-1} = 0$:	If $c_{i-1} = 1$:
a) Define $m = \min(p_{i-1}, 1-p_{i-1})$.	a) Define $m = \min(q_{i-1}, 1-q_{i-1})$.
b) Update $p_i = p_{i-1}(1-m) + m(1-c_i)$.	b) Update $q_i = q_{i-1}(1-m) + m(c_i)$.
c) Leave $q_i = q_{i-1}$ unchanged.	c) Leave $p_i = p_{i-1}$ unchanged.

For each chunk, one of *p* and *q* remains constant, while the other increases or decreases. Including *m* in the computation dampens the change in propensity by its likelihood of inference, so that surprising inferences have a greater effect on propensities than unsurprising inferences. Note that, other than in the pathological case of its initial state, a propensity of 1 is never possible.

Results

Analysis of data is limited to the first three sessions. Protocol 1 is an excerpt from the transcription of the second session, which was divided into four chunks (corresponding to chunks 15-18 in Table 1), each in the task setting of linear, rectangular bars. Following the transcript is a description of coding by context and chunk.

Protocol 1. Portion of Transcription from Session Two

1. *Teacher*: Okay, I'll give you a different question, and this time I'm going to give it to [Charles] first. (Puts out 1 brown bar and 8 red bars). What fraction is 1 red bar out of the brown bar? **[Note that the brown is the length of four red bars]**
2. *Charles*: (Lines 8 red bars above the brown bar). Eight-fourths? An improper fraction? Cause you said, 'the red bar of the brown bar.'
3. *Teacher*: What fraction is one (holding up one finger) red bar out of the brown bar?
4. *Charles*: (Touches brown bar). So that would be four-eighths.
5. *Teacher*: Four-eighths, can you say why (interrupted)
6. *Charles*: That would be the same as one-half.
7. *Teacher*: (To D.J.) What do you think? Do you also think its four-eighths?
8. *D.J.*: Nods.
9. *Teacher*: (Moves 6 red bars and the brown bar to D.J.s desk) What fraction is one red bar of the brown bar now?
10. *D.J.*: (Lines the red bars up, one by one, from left to right above the brown bar)
11. *Charles*: I've got the answer.
12. *D.J.*: One-third. Cause four pieces makes that (the brown) so if there were two it's close, it's like one-half, and if you take that, it's one-third, or something.

13 *Teacher*: Okay, 'cause it's close to one-half?

14. *D.J.*: Yeah, two is close to one-half, so if you take away one it's one-third.
15. *Teacher*: Okay, Charles, what do you think? You said you had the answer, what was

your answer?

16. *Charles*: (Confidently) One-fourth.
17. *Teacher*: Can you show me how you got one-fourth? Show it to me, and show it to D.J.
18. *Charles*: (Lines up all 6 red pieces). This brown bar equals up to four pieces. You said to only count one of the pieces, so that would be one-fourth.
19. *Teacher*: Did that make sense to you D.J.?
20. *D.J.*: It makes more sense than mine.
21. *Teacher*: So you really don't need these other (extra two) pieces do you, to show it?
22. *Charles*: Yeah.
23. *Teacher*: Show it to D.J. so he can see it.
24. *Charles*: See how that equals up to four pieces? So you take these (sweeps away rightmost three) and that's one-fourth.

Coding

The portion of the transcript above was divided into 4 chunks: lines 1 through 2, lines 3 through 8; lines 9 through 14, lines 15 through 24, corresponding with observations of Charles' ecological environment. In the first chunk, I didn't code any teacher actions, whereas in the second chunk, the teacher action was coded as "demonstrating" and "re-iterating," and D.J.'s action was coded as "affirming." In the third chunk, D.J.'s action was coded as "demonstrating" and "explaining" and the teacher's action was coded as "asking for explanation." In the fourth chunk, I coded the teacher's action as "asking for explanation," and "affirming", and I coded D.J.'s action as "affirming." In the first chunk, Charles' uniting of the red bars, counting by ones, and comparing size were his mathematical activity; in the second chunk, Charles' swapping the numerator and denominator and reducing were his mathematical activity. In the third chunk and fourth chunks, I inferred that Charles' was counting by ones, disembedding, and comparing size. In order to clarify my reasoning for making such inferences, I next describe my (retrospective) model of Charles' mathematics.

Hypothetical Model of Charles' Mathematics

In the first two chunks, I infer that Charles was operating with what I call a "proper-placement" scheme, first noticed during the clinical interview, in which he assimilated physical placement (top and bottom) as part of his scheme to achieve a goal of producing a *proper* fraction name. His scheme consisted of the following sequential operations: 1) Make the "top" consist of equivalent marked parts; 2) Count the number of those parts that form the same size as the bottom; 3) Swap the top and bottom if there are more pieces in the top than in the bottom to make the fraction proper; 4) Name the fraction as the number of "top" pieces out of the number of "bottom" pieces. In the latter two chunks, my interpretation is that Charles had modified his scheme to include a disembedding operation, so that even after he united the four red bars to form "four" he was able to operate on the "one" within the "four." As Table 1 indicates, such modification was temporary.

Table 1: First Three Sessions Units Coordination Inferences

Session	Inference of Units Coordination (1 = yes; 0 = no)																		
1	Chunk	1	2	3	4	5	6	7											
	Inference	0	0	0	0	0	0	1											
2	Chunk	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	Inference	0	0	0	0	0	1	1	0	0	1	1	1	1	1	1	1	1	1
3	Chunk	26	27	28	29	30	31	32	33	34	35	36	37						
	Inference	1	1	1	0	0	1	1	1	0	0	0	0						

The Dynamics of Charles' Units Coordination

As the teacher-researcher, I was actively attempting to perturb Charles in subsequent tasks in order to encourage him to use his existing operations to develop a more robust scheme for fractions, i.e., one that would be viable and more useful than those of his "proper-placement" scheme. As seen in Table 1, there were beginnings of inferences of units coordination beginning in chunk 13, followed by successive strings of inferences of units coordination that may have appeared stable to me *during* the teaching sessions.

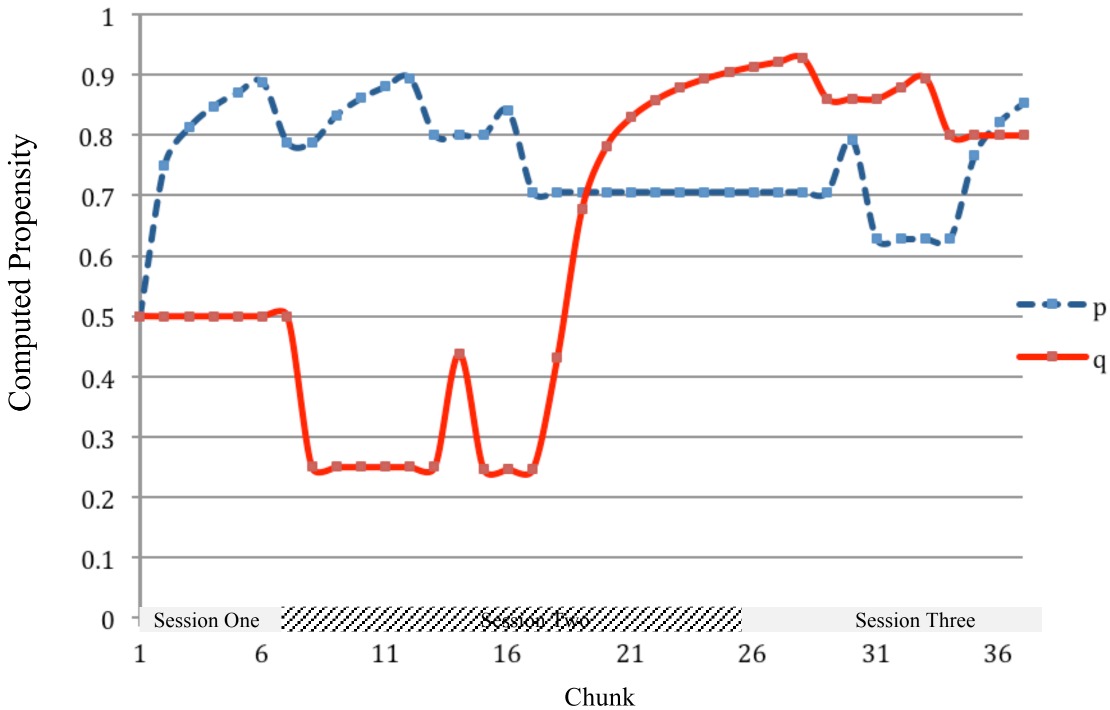

Figure 1. Results of Algorithm 1 with initial equipropensity

The graphs of p and q in Figure 1 illustrate the dynamics of my inferences of Charles' units coordination activity, influenced by my teaching goals. With those caveats, values of p or q converging to 1 indicate lack of perturbation, and a value of q converging to one indicates an accommodation of Charles' assimilatory scheme for fractions to include a disembedding operation.

Discussion

In this proposal, I have presented a third-order stochastic model of the mathematics of a student, as it can be considered a first-order model of my second-order model of Charles' mathematics (Steffe, & Thompson, 2000). It is a mathematization of my conception of the dynamics of the interactions between students' activities, teachers' inferences, and contexts that take place within a goal-oriented constructivist teaching experiment. While I make no claims that it is the *right* way, I believe that it is a useful model because of its compatibility with radical constructivist epistemology (von Glasersfeld, 1995), simplicity, and tractability. In its current form, the model has potential to explicate results stemming from teaching experiment methodology (e.g., to qualify the "emerging" or "establishing" of schemes or to compare students' growth within or across teaching experiments, cf. Simon, Tzur, Heinz, & Kinzel, 2004).

The model could also be used to further validate a theorized role of units coordination in individuals' ways of operating within the domain of fractions and across mathematics content domains.

The presented formulation could be modified to describe or predict the role of context in an individual's scheme development. For example, one might begin with analysis of the contextual factors accompanying large changes in slope in the plots of p or q. Future research might also include multiple teachers/participants and could possibly lead to a predictive or inferential model. Such formulations could eventually generate a dynamic structural model (Fischer, 2006) to simultaneously communicate both an overall structure and the expected variations and deviations from that structure in a way that could more widely influence curricular design and assessment than independent, qualitative comparisons of teaching experiment results (Kilpatrick, 2001).

There are several limitations to this formulation, however. The model only permits dichotomous, un-weighted inferences of students' ways of thinking, but within the existing hierarchy of schemes and associated operations, there are often more than two possibilities, and there are varying degrees of certainty of inference. To test for robustness, i.e., sensitivity to differences in inference or chunking, triangulation of the coding of data is necessary; such qualitative research is quite time-intensive. However, at minimum, the model provides a novel mechanism for communicating the dynamics of the non-linear assimilation/accommodation process - the trajectory and variation *within* a second-order model of a student's mathematics.

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